Recently I was challenged to solve the following problem:

1 Problem: Find the Potential Energy of a Longitudinal wave in an elastic rod

(I solve the problem by two different ways then I relate it to Problem 7-24 French[2] and there are some final remarks about the general properties of wave phenomena)

We consider an elastic rod that changes its length via an incoming longitudinal wave as in the following figure shows,

1.1 Solution 1:

*Hint:* Consider a simpler problem: find the potential energy of a clamped elastic rod of total length $L$ extended a distance $\Delta L$.

Now we will use the hint, let’s consider an elastic rod of length $L$ whose length is changed $\Delta L$ by an external force $F$, it is shown in the following figure:
First of all, we are considering a non-rigid rod. Also this rod have the characteristic that is elastic. This is not just a fancy name, mathematically means that the ratio of the two following quantities is constant

\[ \frac{\text{Stress}}{\text{Strain}} = Y = \text{const.} \] (1)

Here \( S \) stands for the cross sectional area of the elastic rod. Now the ratio of this two quantities is defined by \( Y \), this is the Young’s Modulus and if the extension of the rod is sufficiently small \( Y = \text{const.} \).

Now let’s compute the work done that the force \( F \) has to do to extend this rod a distance \( x \), using the above relations,

\[ dW = F dx = Y S \left( \frac{x}{L} \right) dx. \]

Now the total work is found integrating from 0 to \( \Delta L \):

\[ W = \int_0^{\Delta L} Y S \left( \frac{x}{L} \right) dx = \frac{1}{2} Y S L \left( \frac{\Delta L}{L} \right)^2. \] (2)

Now we consider the more difficult problem, if we look at the first figure we have a differential element of the elastic rod whose initial length is \( \Delta x \) and the passing wave make it extend a distance \( \Delta \xi \).

Essentially we have the same situation that we just solved (the potential energy to extend a differential rod \( \Delta x \) a distance \( \Delta \xi \)), we have the following analogies:

\[ \Delta L \rightarrow \Delta \xi, \quad L \rightarrow \Delta x, \]

thus for a differential element of an elastic rod we have:

\[ \Delta W = \frac{1}{2} Y S \Delta x \left( \frac{\Delta \xi}{\Delta x} \right)^2, \]

\[ \frac{\Delta U}{\Delta x} = \frac{1}{2} Y S \left( \frac{\partial \xi}{\partial x} \right)^2. \]

Now we can find the potential energy density per unit length \( \frac{dU}{dx} \) taking the limit as \( \Delta x \searrow 0 \):

\[ \frac{dU}{dx} = \frac{1}{2} Y S \left( \frac{\partial \xi}{\partial x} \right)^2. \] (3)

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\(^1\)Even if the extension of the rod is in the \( x \) direction only, we are still considering a 3-dimensional bar, this stood out when we introduced the cross sectional area \( S \).
By taking the limit we find the values of the potential energy density $\frac{dU}{dx}$ and strain $\frac{\partial \xi}{\partial x}$ at a particular point $x$ of the rod.

### 1.2 Solution 2:

There is another rather surprising solution to this problem, it is the usual approach to introduce the *Lagrangian Field Theories* [1]. The procedure is to find the potential energy for a discrete system of equal masses connected by springs as in the following picture and then take the continuum limit as the distance between the masses $a \to 0$.

The conclusions of the following reasoning doesn’t change if the discrete system has $N$ masses (the total length of the system is $A = Na$) or if the system is extended infinitely in any direction. The important fact is that the distances between the masses is always $a$, additionally for the finite length system we impose that $N \to \infty$.

Now the total potential energy of the system is:

$$U = \frac{1}{2} \sum_i k(\xi_{i+1} - \xi_i)^2.$$  

Where it’s clear that the energy of every single spring is a function of its displacement $(\xi_{i+1} - \xi_i)$ from the equilibrium position. If we were to have a system with $N$ masses the validity of the previous result would depend on the boundary conditions of the system. If we can displace freely the first and last masses, then this energy is exact.

Now we will write the last energy conveniently in order to take the continuum limit,

$$U = \frac{1}{2} \sum_i ka\left(\frac{\xi_{i+1} - \xi_i}{a}\right)^2 a. \quad (4)$$
We can easily see that as we take the limit, the discrete index $i$ becomes a continuous index $x$ and the following quantities from the energy transform to:

$$a \rightarrow 0 \quad \int dx \quad \frac{\xi_{i+1} - \xi_i}{a} \rightarrow \Delta x \quad \frac{\xi(x + \Delta x) - \xi(x)}{\Delta x} \rightarrow \frac{\partial \xi}{\partial x}.$$

Now we want to know what happens with the term $ka$ as we take the limit $a \rightarrow 0$, does it even make sense to ask $ka \rightarrow 0$?

It turns out that in order to get to the right model we have to impose $^{2}$

$$ka \rightarrow SY.$$

However this is not intuitive as the others limits are, at first one may think that $ka$ approaches to something really small. Though this seems appealing, it yields a wrong theoretical conclusion.

**UPDATE**$^{3}$ (Mostly adapted from [5]):

One can clearly see how a quantity like $\mu = m/a$ (mass density per unit length) would yield a finite result in the refinement process since one expects both the mass $m$ and the lattice spacing $a$ to decrease when we go to very small length-scales.

However for the quantity $ka$ to yield a finite result, each spring proper stiffness $k$ would become necessarily stronger and stronger as the lattice refinement process proceeds: $k(a) \rightarrow \infty$ as $a \rightarrow 0$. This is the problematic concept !!, however as we'll see in the refinement process, the springs $k$ don't add up cumulative to an effective constant $K$, but they add in series (like parallel resistors) as opposed to our intuition.

To see this lets consider the case of a finite spring chain with stiffness $k$ and $N$ masses,

As we are considering a uni-dimensional system using a spring chain model the correct result is $ka \rightarrow Y$ [1](pg. 559), however we consider a 3-dimensional generalization because of the preceding results of linear elasticity theory, (1).

One day browsing the web I found the notes [5], to my surprise I discovered this remarkable derivation in a somehow different notation. All the lecture notes are really good, take a look.
Clearly if the total length of the spring chain (length between the two barriers) is $l$ and the length between the masses in equilibrium is $a$, then we have

$$(N + 1)a = l, \quad M = Nm,$$

where $M$ is the total mass of the spring chain.

We can rewrite the previous expressions as,

$$N = \frac{l}{a} \left(1 - \frac{a}{l}\right), \quad m = \frac{M}{l} a \left(1 - \frac{l}{a}\right)^{-1} = \mu a \left(1 - \frac{l}{a}\right)^{-1},$$

where $\mu = M/l$ is the linear mass density of the spring chain.

So what we are going to impose is that in the refinement $a \downarrow 0$ the quantity $\mu$ keeps constant, the same in the limiting case of a compressional wire (or "string") as for the spring chain from which we started. This implies that the total number of springs $N$ have to increase like $\mathcal{O}(a^{-1})$ and the masses $m$ of each of them have to decrease as $\mathcal{O}(a)$. To see this clearly when $a \ll l$ in the previous equations,

$$N = \frac{l}{a}, \quad m = \mu a.$$

From this equations, we can see that since $\mu$ is kept constant in the limiting process, the value of the individual masses $m$ are going to decrease in the refinement. This allows us to approximate the spring chain when $a \ll l$ as a chain of spring in the so-called series configuration, i.e. a series of springs connected by massless contacts.

So if we have $n$ springs of stiffness $k_i$ each, the series configuration of springs would have a total or effective stiffness $k_T$ of,

$$\frac{1}{k_T} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \ldots + \frac{1}{k_n}.$$

So if we have a series configuration of springs of length $l$ and effective stiffness $K = SY/l^4$ that have been assembled by connecting in series $N + 1$ identical springs $k$ of length $a = l/(N + 1)$ we have,

$$\frac{1}{K} = \frac{l}{SY} = N \frac{1}{k} \quad \text{with} \quad N = \frac{l}{a}, \quad \text{when} \quad a \ll l.$$

Then,

$$k = N \frac{SY}{l} = \frac{SY}{a} \quad \text{when} \quad a \ll l.$$
This implies that in the refinement process \( a \downarrow 0 \) the quantity,

\[
ka \longrightarrow SY,
\]

where we are imposing \( SY \) also as a constant in the refinement process.

So we checked our initial claim that the springs \( k \) become necessarily stronger and stronger as the lattice refinement process proceeds: \( k(a) \not\rightarrow \infty \) as \( a \downarrow 0 \). However since the springs add in series they manage to generate a constant effective spring stiffness \( K = SY/l \) in the refinement process, because in the refinement as the spring constant \( k \) grows as \( \mathcal{O}(a^{-1}) \) the number of springs \( N \) grows also as \( \mathcal{O}(a^{-1}) \), then since \( K = N/k \), the effective spring stiffness remains constant.

With this result we can take the limit of the energy (4), this yields,

\[
U = \frac{1}{2} \sum_i ka \left( \frac{\xi_{i+1} - \xi_i}{a} \right)^2 a \longrightarrow \frac{1}{2} \int SY \left( \frac{\partial \xi}{\partial x} \right)^2 dx,
\]

where we used all the previous results and the sum became an integral in the limiting to the continuum.

Gathering all the previous results we obtain the total potential energy:

\[
U = \frac{1}{2} \int SY \left( \frac{\partial \xi}{\partial x} \right)^2 dx.
\]

So we obtained again by a rather different conceptually different method the potential energy density

\[
\frac{dU}{dx} = \frac{1}{2} Y S \left( \frac{\partial \xi}{\partial x} \right)^2.
\] (5)

For an application of this result I found the following problem in the book [2]:

2 Problem 7-24 French

Consider a longitudinal sinusoidal wave \( \xi = \xi_0 \cos \left( 2\pi k(x - vt) \right) \) travelling down a rod of mass density \( \rho \), cross-sectional area \( S \) and Young’s modulus \( Y \). Show that if the stress in the rod is due to solely the presence of the wave, the kinetic energy is \( k_x = \frac{1}{2} \rho S \left( \frac{\partial \xi}{\partial t} \right)^2 \), and the potential energy \( u_x = \frac{1}{2} Y S \left( \frac{\partial \xi}{\partial x} \right)^2 \). Thus show that the kinetic energy per wavelength and the potential energy per wavelength both equal \( \frac{1}{4} \left( \rho S \lambda \right) z_0^2 \), where \( z_0 \) is the maximum particle velocity \( \left( \frac{\partial \xi}{\partial t} \right) \).
Solution:

We already showed that the Potential Energy density solely to the presence of a Longitudinal wave in an elastic rod is:

\[
\frac{dU}{dx} = \frac{1}{2} YS \left( \frac{\partial \xi}{\partial x} \right)^2. \tag{6}
\]

Now the potential energy in a wavelength of the wave is:

\[
U_\lambda = \int_0^\lambda \frac{1}{2} YS \left( \frac{\partial \xi}{\partial x} \right)^2 dx,
\]

\[
= \frac{1}{2} YS z_0^2 \int_0^\lambda \frac{(2\pi)^2}{\lambda^2} \sin \left( 2\pi k(x - vt) \right)^2 dx,
\]

\[
= \frac{1}{2} YS z_0^2 T^2 \frac{1}{\lambda^2} \left[ \int_0^\lambda \sin \left( 2\pi k(x - vt) \right)^2 dx \right],
\]

\[
= \frac{1}{2} YS z_0^2 \frac{\lambda^2}{\sigma^2} \frac{1}{\lambda^2} \left[ \frac{1}{2} \lambda \right],
\]

\[
= \frac{1}{4} YS z_0^2 \frac{1}{v^2} \lambda.
\]

In this derivation we used the relations (French’s notation) \( T = \frac{1}{\omega}, \lambda = \frac{1}{k} \) and we also used that max(\( \frac{\partial \xi}{\partial t} \)) = \( z_0 \). Now we know that the velocity \( v \) of a wave in an elastic rod is \( v^2 = \frac{Y}{\rho} \), so this yield the desired result:

\[
U_\lambda = \frac{1}{4} (\rho S \lambda) z_0^2. \tag{7}
\]

It’s straight forward to show that for a mass element \( \Delta m = \rho S \Delta x \) the kinetic energy density is:

\[
\frac{dK}{dx} = \frac{1}{2} \rho S \left( \frac{\partial \xi}{\partial t} \right)^2, \tag{8}
\]

Now the kinetic energy in a wavelength of the wave is:
\[ K_\lambda = \int_0^\lambda \frac{1}{2} \rho S \left( \frac{\partial \xi}{\partial t} \right)^2 dx, \]
\[ = \frac{1}{2} \rho S \xi_0^2 \left[ \int_0^\lambda \sin \left( 2\pi k(x - vt) \right)^2 dx \right], \]
\[ = \frac{1}{2} \rho S \xi_0^2 \left[ \frac{\lambda}{2} \right]. \]

**Note 1** We can see that the total energy per wavelength, \( E_\lambda = U_\lambda + K_\lambda \), yields:
\[ E_\lambda = \frac{1}{2} (\rho S \xi_0^2). \]

This can be written using the relations: \( \max(\frac{\partial \xi}{\partial t}) = \xi_0 = \frac{\xi_0 2\pi}{ct} \) and \( T = \frac{1}{\omega} \):
\[ \frac{E_\lambda}{\lambda} = \frac{1}{2} (\rho S 4\pi^2 \xi_0^2 \omega^2). \] (9)

This relation underlies a somehow general principle of wave phenomena, the energy density is proportional to the square of the amplitude and frequency, this can be seen in:

1. Transverse waves in a string
2. Longitudinal waves in bars
3. Acoustic Sound waves

I have not studied the other wave phenomena in detail, but here goes some good references where the reader can get deeper in this claims:

**References**